Section 9.1

Section 9.2 Direct Products

<u>Recall:</u> Th 7.4 Let G and H be groups

$G\times H = \{(g,h) \mid g\in G, h\in H\}$ is a group under the operation

$$(g_1,h_1)(g_2,h_2) = (g_1g_2, h_1h_2)$$

$G\times H$ is the direct product of G and H

<u>Prop</u> Let $G = M\times N$ (M and N are groups)

1. Then G has two normal subgroups, call them M' and N' such that $M'\simeq M$ and $N'\simeq N$.
2. $M'\cap N' = e_G = (e_M, e_N)$
3. $G = M'N' = \{mn \mid m\in M', n\in N'\}$

<u>Pf</u> Let $M' = \{(m, e_N) \mid m\in M\}$, $N' = \{(e_M, n) \mid n\in N\}$ $\Big\}$ both are subgroups of G $\quad M'\simeq M$, $N'\simeq N$

<u>remark</u> $(m,e_N)(e_M,n) = (m,n) = (e_M,n)(m,e_N)$

1. The map $G = M\times N \longrightarrow N$, $(m,n)\mapsto n$ is a surjective group homomorphism; its kernel is $\{(m,e_N) \mid m\in M\} = M'$, therefore M' is normal

$$G = M \times N \to M$$
$$(m, n) \mapsto m$$

N' is normal

② $M' \cap N' = \{(e_M, e_N)\} = \{e_G\}$

③ $M'N' = \{ab \mid a \in M', b \in N'\} \subseteq G$ because $a \in M' \subset G$, $b \in N' \subset G$ thus $ab \in G$

Every element of G appears in $M'N'$:

$(m, n) = (m, e_N)(e_M, n)$ thus $M'N' \supseteq G$ Therefore $M'N' = G$.

The conditions ①, ②, ③ are sufficient to conclude that a group is (isomorphic to) a direct product of two subgroups of the group

Th 9.3 Let M and N be normal subgroups of G.

Assume that $M \cap N = \{e\}$,

$$G = MN = \{mn \mid m \in M, n \in N\}$$

Then $G \simeq M \times N$

Easy to check

$M \times N \simeq N \times M$

$(m, n) \longmapsto (n, m)$

Rem If the theorem is true, then $MN = NM$

Furthermore (see a remark above) it should be that $mn = nm$ for any $m \in M$, $n \in N$

Lemma 9.2 (Exer 21, p 254)

Let M and N be normal subgroups in G such that $M \cap N = \{e\}$.

Then for any $m \in M$ and $n \in N$, we have that $mn = nm$.

Pf. Consider $m\overbrace{nm^{-1}n^{-1}}^{\in M} \in N \cap M = \{e\}$

N is normal in G, thus $mNm^{-1} = N$ for every $m \in G$, in particular, for $m \in M \subset G$

Thus, for $n \in N$ we have $mnm^{-1} \in N$

We conclude that for any $m \in M$ and $n \in N$,

$$mnm^{-1}n^{-1} = e \qquad mnm^{-1} = n \qquad mn = nm$$

Pf (Thm 9.3)

Define a map $f: M \times N \longrightarrow G$ Wanted: f is an isomorphism

$(m, n) \mapsto mn$

① f is surjective because $MN = \{mn \mid m \in M, n \in N\} = G$

② f is a homomorphism

$f((m_1, n_1)(m_2, n_2)) = f((m_1, n_1)) f((m_2, n_2))$ - to check

$(m_1, n_1)(m_2, n_2) = (m_1 m_2, n_1 n_2)$

$\underline{m_1 m_2 n_1} n_2 = m_1 n_1 m_2 n_2$ - to check

Lemma 9.2: $n_1 m_2 = m_2 n_1$

The right side becomes

$\underline{m_1 m_2 n_1} n_2$

③ f is injective

it suffices to check that the kernel of f is trivial

If $f((m, n)) = e$, then $mn = e$ that is

$$M \ni m = n^{-1} \in N$$

$$n \in N$$

If $m = n^{-1}$, then $m = n^{-1} \in M \cap N = \{e\}$. Thus $m = n = e$.

Thus the kernel of f is trivial.

For a group to be (isomorphic to) a direct product of several subgroups

Th 9.1 Let $N_1, \ldots, N_k$ be normal subgroups of G.

Assume that every element $g \in G$ can be written as

$$g = a_1 \cdots a_k \quad \text{with } a_i \in N_i$$

in a unique way

Then $G \simeq N_1 \times \cdots \times N_k$

Pf

Let $f : N_1 \times \cdots \times N_k \longrightarrow G$ Wanted: f is an isomorphism

$$(a_1, \ldots, a_k) \longmapsto a_1 \cdots a_k$$

f is surjective because any $g \in G$ can be so written

f is injective because ______ / ______ in a unique way

f is a group homomorphism

By Lemma 9.2, elements from distinct N_i commute $\Leftarrow$

Claim $N_i \cap N_j = \{e\}$ if $i \neq j$

If $a \in N_i \cap N_j$ then the uniqueness fails:

$a = e \cdots \underset{i}{a} \cdots e = e \cdots \underset{j}{a} \cdots e$

To check: $f((a_1, \dots a_k)(b_1, \dots b_k)) = f((a_1, \dots, a_k)) f((b_1, \dots, b_k))$

$$a_1 b_1 a_2 b_2 \dots a_k b_k = a_1 a_2 \dots a_k b_1 b_2 \dots b_k$$

checked

Lemma 9.2 allows us to do these moves